U.S. DEPARTMENT OF COMMERCE
National Technical Information Service

AD-A036 687

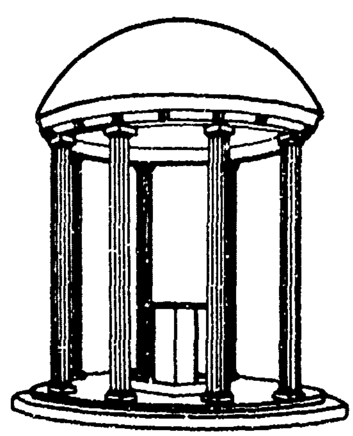
VARIANCE REDUCTION TECHNIQUES FOR NONSTATIONARY SIMULATION MODELS

NORTH CAROLINA UNIVERSITY AT CHAPEL HILL

FEBRUARY 1977

OPERATIONS RESEARCH AND SYSTEMS ANALYSIS

UNIVERSITY OF NORTH CAROLINA
AT CHAPEL HILL



VARIANCE REDUCTION TECHNIQUES FOR NONSTATIONARY SIMULATION MODELS

George S. Fishman

Technical Report No. 77-3

Curriculum in Operations Research
and Systems Analysis
University of North Carolina at Chapel Hill

This research was supported by the Office of Naval Research under contract N00014-76-C-0302.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 77-3	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Variance Reduction Techniques for Nonstationary Simulation Models		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) George S. Fishman		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0302
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of North Carolina Chapel Hill, N.C. 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Operations Research Program Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE February 1977
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear Programming · Population Growth · Simulation
Nonstationarity · Variance Reduction

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Most variance reduction techniques encountered in simulations are designed to operate with stationary models. However, many simulations are nonstationary in character, population growth models being an example. One way to facilitate statistical inference in a nonstationary simulation is to interchange the order of replication collection and time evolution.

That is, at each time point several replications are performed to enable a user to estimate parameters at that point in time. This paper describes three variance reduction techniques that use this interchange between collection and evolution to induce negative correlation between replications, thereby producing estimates with smaller variances. Model 1 describes a procedure that occasionally relies on the solution of a linear program to develop an optimal sampling plan. Model 2 offers an alternative that applies when the populations in strata are large. Model 3 applies when survival probabilities are functions of an exogenous random variable such as rainfall. A female elephant population simulation illustrates the success one can expect with model 1.

Abstract

Most variance reduction techniques encountered in simulations are designed to operate with stationary models. However, many simulations are nonstationary in character, population growth models being an example. One way to facilitate statistical inference in a nonstationary simulation is to interchange the order of replication collection and time evolution. That is, at each time point several replications are performed to enable a user to estimate parameters at that point in time. This paper describes three variance reduction techniques that use this interchange between collection and evolution to induce negative correlation between replications, thereby producing estimates with smaller variances. Model 1 describes a procedure that occasionally relies on the solution of a linear program to develop an optimal sampling plan. Model 2 offers an alternative that applies when the populations in strata are large. Model 3 applies when survival probabilities are functions of an exogenous random variable such as rainfall. A female elephant population simulation illustrates the success one can expect with model 1.

1. Introduction

Among the tools available to the simulation user, variance reduction techniques have from their inception held considerable attraction. Broadly speaking a variance reduction technique is a sampling plan that enables a user to achieve a specified level of statistical accuracy for less cost than pure random sampling permits or, conversely, it allows a user to realize greater accuracy for specified cost. Hammersley and Handscomb [2] describe variance reduction techniques applicable in Monte Carlo simulation. Fishman [1], Kleijnen [3] and Mihram [4] describe the extension of these methods to discrete event digital simulation.

In reviewing the techniques and the problems for which they are appropriate one notes an emphasis on their use in simulations of stationary models. A stationary model describes an environment in which means and covariance structures remain constant in time. However, the simulation method also applies to nonstationary systems of which population growth models form a substantive share. In a population model each year or period begins with a frequency distribution of population by stratum based on age sex, etc. In advancing time by one period the simulation exposes each member of each stratum to the risk of death so that at the end of the period a new distribution by strata appears. If the expected frequency in each stratum is constant through time, then stationarity prevails. However, this stationarity requires a fortuitous balance between birth rates, age specific

†This work was motivated by discussions on the construction of population simulations for long-lived species with Professor Daniel Botkin of the Ecosystems Center of the Marine Biological Laboratory at Woods Hole, Massachusetts and Professors Richard Miller and Matthew Sobel of Yale University. I am grateful to them for introducing me to the problem and for their valuable insights. I am also grateful to William Kwapil and Mark Miller for providing computational assistance in preparing Tables 1 and 2.

survival rates and other rates More commonly, the shape of the strata frequency distribution, the mean number of members in the population or both change as time evolves so that nonstationarity prevails.

Now the difficulty with nonstationarity in practice is that one cannot apply the statistical methods developed for analyzing simulation output for stationary models. These latter techniques take two principal forms. The first analyzes data within a single run to estimate parameters of interest that are time independent and the second uses averages formed on individual replications to estimate the parameters [1]. Because nonstationarity implies time dependent parameters one has to look elsewhere for analysis techniques. One approach is to use the data on sample frequency distributions collected on several replications at the same time point to estimate the true frequency distribution at that time. This technique is legitimate, provided that all replications begin with the same initial strata frequency distribution and population.

Analysis across replications can proceed in at least two ways. One way has the simulation user run replications serially, store the sample frequency data at each time point in each run and then combine corresponding data, after all runs are completed, to estimate the mean strata frequency distribution at each point. For a model with many strata run for a long period of time the raw data may require substantial storage. Moreover, the simulation user must wait for completion of all runs before learning about interim estimates.

In an alternative method of analysis that we plan to use here, the simulation user performs replications at a given time point before proceeding to the next time. In effect, he reverses the order of time advance and replication. Doing so enables him to estimate the mean strata frequency distribution at a given time point before moving forward. If an online

graphics device is available, looking at successive images of this estimated distribution may provide sufficient information to terminate the simulation prematurely. Or the information gained may encourage the user to change certain parameters of the simulation to see how the population responds.

Although the display and interpretive feature alone make this second alternative attractive, the ability to apply a variance reduction te nnique to corresponding strata in the different replications at a given time point offer an additional attraction. Especially since the accuracy achieved for a given cost can be considerably greater than independent replications allow.

In this paper we describe three methods of variance reduction applicable to population growth models. Model 1 in Section 2 outlines a method appropriate for a simulation in which the probability of survival in each strata in each year randomly assumes one of two possible values. As n the number of replications increases, the achievable variance reduction often increases dramatically. Occasionally the technique calls for the solution of a moderate size linear program, perhaps an impediment today but not necessarily so tomorrow.

Section 3 describes a second variance reduction technique that applies for the survival probability mechanism of model 1 when the number in each stratum is large. This technique appears most relevant when the two possible survival probabilities per strata are close in value. Section 4 offers a third variance reduction technique that applies when the survival probability is a continuous function of an exogenous variable such as rainfall which itself follows a normal probability law. To give the reader an appreciation of potential performance Section 5 describes the application of model 1 in Section 3 to a simulation of female elephant population growth. The results are encouraging.

2. Model 1

Let W denote a random variable whose distribution is rectangular on $(0,1)$. Then W defines a uniform deviate and $U(0,1)$, its distribution. Let $B(N,\theta)$ denote the binomial distribution with integral parameter N and probability θ. Define X and Y as random variables from $B(N,\alpha)$ and $B(N,\beta)$, respectively, and let I denote the binary random variable

$$(1) \qquad I = \begin{cases} 1 & 0 \le W \le p \\ 0 & p < W < 1 \end{cases} .$$

To illustrate the use of these concepts, assume that in a given year a member of a particular age stratum of a population has a survival probability α with probability p or β with probability $1-p$. For example, in an animal population the different survival rates arise as a consequence of favorable (α) or unfavorable (β) weather conditions. Then one expects $\alpha>\beta$. If the stratum has N members then the number who survive to enter the next age stratum the next year is

$$(2) \qquad R = IX + (1-I)Y = I(X-Y) + Y .$$

Define the operator Δ to mean deviation from expectation. Then

$$(3) \qquad \Delta R = N(\alpha-\beta)\Delta I + p\,(\Delta X - \Delta Y) + \Delta I(\Delta X - \Delta Y) + \Delta Y$$

which, provided W is independent of X and Y, has mean zero and variance

$$(4) \qquad V = \text{var}\,(R) = p(1-p)\,N^2\,(\alpha-\beta)^2 + N[\alpha(1-\alpha) + (1-p)\beta(1-\beta)].$$

Applying the procedure to each stratum with its corresponding p, N, α and β, one can construct a sample age distribution for the year following the one in which sampling occurred. Repeated application through successive years allows one to see how the sample age distributions change over time.

Since our principal interest concerns nonstationary features of the distribution, it is important that the variance of the ordinates in each

sample age distribution in each year remain within acceptable limits for meaningful inference. If in each year one performs n independent replications for each age stratum, then the sample mean ordinate for a given stratum has variance V/n, V being determined by the stratum. The objective of the present research is to devise a sampling plan among the n replications that not only produces a smaller variance but actually produces the smallest possible for n replications. Although an attempt at variance reduction is always desirable, it assumes particular importance in the present case. Notice that the leading term in (4) is proportional to N^2, not N as would occur in the binomial case ($\alpha=\beta$). This source of variation provides additional impetus in the search for a sampling plan that will lead to acceptable estimates of the ordinates of an age distribution.

Let a subscript on I, N, X, Y, V and W denote replication number. Define W_i as

(5)
$$W_i \equiv \theta_i + U \pmod 1$$

$$0 \le \theta_i \le 1 \qquad i = 1,\dots,n$$

so that $W_1,\dots, W_n$ are uniform, but not independent, deviates. Then for a particular age stratum replication i yields

(6)
$$R_i = I_i\,(X_i - Y_i) + Y_i$$

(7)
$$\Delta R_i = N_i\,(\alpha-\beta)\Delta I_i + p(\Delta X_i - \Delta Y_i) + \Delta I_i\,(\Delta X_i - \Delta Y_i) + \Delta Y_i\ .$$

Here R_i denotes the number of survivors on replication i and

$$\bar{R}_n = n^{-1} \sum_{i=1}^{n} R_i$$

denotes the sample mean number of survivors over the n replications. If these replications are independent then $\bar{R}_n$ has variance V/n.

Consider the case in which $X_1, \ldots, X_n$ are i.i.d., $Y_1, \ldots, Y_n$ are i.i.d. and $\{X_i\}$ and $\{Y_i\}$ are independent. Then $\bar{R}_n$ has variance

(8) $$\mathrm{var}^{(1)}(\bar{R}_n) = (V_n^{(1)} + 2U_n^{(1)})/n$$

(9) $$nV_n^{(1)} = p(1-p)(\alpha-\beta)^2 \sum_{i=1}^{n} N_i^2 + [p\alpha(1-\alpha) + (1-p)\beta(1-\beta)] \sum_{i=1}^{n} N_i$$

(10) $$nU_n^{(1)} = (\alpha-\beta)^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} N_i N_j \operatorname{cov}(I_i, I_j)$$

where the superscript $^{(1)}$ denotes the model number. Let

$$(\phi)^+ \equiv \max(0,\phi) .$$

Since a little thought shows that for $\{\theta_i \geq \theta_{i-1},\ i=2,\ldots,n\}$

(11) $$E(I_i I_j) = (p-\theta_j)^+ + (p-1-\theta_i+\theta_j)^+ + (p+\theta_i-\theta_j-(p-\theta_j)^+)^+ \qquad j>i,$$

one can show that the optimal $\theta_1, \ldots, \theta_n$ for minimizing $U_n^{(1)}$ emerge from the solution of the linear program:

(12) $$C_n^{(1)} = \min_{\underline{\theta}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} N_i N_j (D_j + E_{ij} + F_{ij}) \qquad \underline{\theta} \equiv (\theta_1, \ldots, \theta_n)$$

subject to the $n(n+1)$ constraints

(13)
$$\theta_1 \geq 0$$
$$\theta_n \leq 1$$
$$\left.\begin{array}{l} \theta_j \geq \theta_{j-1} \\ D_j \geq p-\theta_j \end{array}\right\} \quad j = 2, \ldots, n$$

$$\left.\begin{array}{l} E_{ij} \geq p-1-\theta_i+\theta_j \\ \\ F_{ij} \geq p+\theta_i-\theta_j-D_j \end{array}\right\} \quad j = i+1,\ldots, n; \quad i=1,\ldots, \quad n-1$$

with α, β, $N_1,\ldots, N_n$ specified. Notice that the optimal $\underline{\theta}$ is independent of α and β.

To measure the extent of variance reduction with a particular model, one can use

(14) $$\omega_n^{(k)} = V_n^{(k)}/n \min_{\underline{\theta}} [\mathrm{var}^{(k)}(\bar{R}_n)]$$

The quantity $n\omega_n^{(k)}$ denotes the number of replications that independent sampling requires to achieve the same variance that results from using the optimal sampling plan for model k. For model 1 one has

(15) $$\omega_n^{(1)} = 1/\left\{1 + 2(\alpha-\beta)^2 [C_n^{(1)} - p^2 \sum_{i=1}^{n-1} N_i \sum_{j=i+1}^{n} N_j] /V_n^{(1)}\right\}$$

which for large $N_1,\ldots, N_n$ is

(16) $$\omega_n^{(1)} \sim 1/[1 + 2(C_n^{(1)} - p^2 \sum_{i=1}^{n-1} N_i \sum_{j=i+1}^{n} N_j)/p(1-p) \sum_{i=1}^{n} N_i^2]$$

$$\cdot (1-p)/(1-np + 2C_n^{(1)} /N^2np)$$

where the upper bound obtains when $N_1 = N_2 = \ldots = N_n = N$.

Since large $N_1,\ldots, N_n$ lead to a large variance for $\bar{R}_n$ when using independent replications, we explore this case in more detail. In particular this allows us to ignore linear terms in N_i in ( ). Suppose that $p \leq 1/n$. If $\theta_i = (i-1)\rho$, where $p \leq \rho \leq 1/n$ for $i=1,\ldots, n$, then $\mathrm{cov}(I_i, I_j) = -p^2$ so that $C_n^{(1)} = 0$ and

(17) $$\omega_n^{(1)} \sim 1/[\,1-2p \sum_{i=1}^{n-1} N_i \sum_{j=i+1}^{n} N_j/(1-p)\sum_{i=1}^{n} N_i^2]$$

$$= (1-p)/[1-p(\sum_{i=1}^{n} N_i)^2/\sum_{i=1}^{n} N_i^2] \le (1-p)/(1-np) \;.$$

This case avoids the need to solve the linear program in (13) and (14). Moreover it applies equally well for $p \geq 1 - 1/n$ since one can substitute $(1-p)$, β and α for p, α and β, respectively, without loss of generality.

Solutions of the linear program for $1/n < p < 1/2$ with $N_1 = \ldots = N_n = N$ also yield worthwhile insights. Perusal of Table 1 for $i/n < p \le (i+1)/n$, $i = 1,\ldots,\lfloor n/2-1 \rfloor$ showed that for large N

(18) $$\text{var}^{(1)}(\bar{R}_n) = N^2\,(\alpha-\beta)^2\,q(1-nq)/n$$

for $q = p-i/n$ so that

(19) $$\omega_n^{(1)} = p(1-p)/q(1-nq) \;.$$

The variance in (18) vanishes for $q=0$ and $1/n$ and has a maximum $N^2\,(\alpha-\beta)^2/4n^2$ for $q = 1/2n$. This implies that

(20) $$\min_{\underline{\theta}} \text{var}^{(1)}\,(\bar{R}_n) \le N^2(\alpha-\beta)^2\,/4n^2 \qquad 0 < p < 1/2$$

so that

(21) $$\omega_n^{(1)} \geq 4n\,p(1-p) \;.$$

For examples, $n=2$ and $p=.25$ leads to $\omega_n^{(1)} \geq 1.5$ whereas $n=10$ and $p=.4$ leads to $\omega_n^{(1)} \geq 9.6$.

Table 1

Optimal θ's for Selected n and p

$N_1 = N_2 = \ldots = N_n$

n	p	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	$\frac{p(1-p)}{n}$	$\frac{var^{(1)}(R_n)}{(a-b)^2}$	Variance Reduction	$n^{*\dagger}$
3	.35	.30	.65	1.00								.0758	.0053	14.36	3
	.40	.20	.60	1.00								.0800	.0178	6.75	3
	.45	.10	.55	1.00								.0825	.0253	3.2634	3
	.50	.00	.50	1.00								.0833	.0833	1.000	2
4	.30	.00	.30	.60	.90							.0525	.0100	5.25	4
	.35	.00	.30	.65	.95							.0539	.0150	3.79	4
	.40	.00	.20	.60	.80							.0600	.0150	4.00	4
	.45	.00	.10	.55	.55							.0619	.0100	6.19	3
	.50	.00	.00	.50	.50							.0625	.0000	-	2
5	.25	.00	.25	.25	.50	.75						.0375	.0075	5.00	4
	.30	.00	.30	.30	.60	.90						.0420	.0100	4.20	4
	.35	.00	.05	.35	.65	.70						.0455	.0075	6.07	5
	.40	.00	.20	.40	.60	.80						.0480	.0000	-	5
	.45	.00	.10	.20	.55	.65						.0495	.0075	6.60	5
	.50	.00	.00	.00	.50	.50						.0500	.0100	5.00	2
6	.20	.00	.20	.20	.40	.60	.80					.0267	.0044	6.00	5
	.25	.00	.25	.25	.50	.75	1.00					.0313	.0069	4.50	4
	.30	.00	.00	.30	.30	.60	.70					.0350	.0044	7.88	4
	.35	.00	.00	.30	.35	.65	.70					.0379	.0025	15.17	5
	.40	.00	.03	.20	.40	.60	.80					.0400	.0067	6.00	5
	.45	.00	.00	.10	.45	.55	.55					.0413	.0058	7.07	4
	.50	.00	.00	.00	.50	.50	.50					.0417	.0000	-	2
7	.15	.00	.15	.30	.40	.55	.70	.85				.0182	.0020	8.93	7
	.20	.00	.00	.20	.20	.40	.60	.80				.0229	.0049	4.67	5
	.25	.00	.00	.25	.25	.50	.75	.75				.0268	.0038	7.00	4
	.30	.00	.10	.30	.40	.60	.70	.90				.0300	.0018	16.33	7
	.35	.00	.00	.30	.35	.60	.65	.95				.0325	.0051	6.43	6
	.40	.00	.00	.20	.40	.40	.60	.80				.0343	.0033	10.50	5
	.45	.00	.10	.20	.45	.55	.65	.90				.0354	.0026	13.86	7
	.50	.00	.00	.00	.50	.50	.50	1.00				.0357	.0051	7.00	2
8	.15	.00	.10	.15	.25	.40	.55	.70	.85			.0159	.0025	6.38	8
	.20	.00	.20	.20	.40	.40	.60	.80	1.00			.0200	.0038	5.33	5
	.25	.00	.00	.25	.25	.50	.50	.75	.75			.0234	.0000	-	4
	.30	.00	.00	.20	.30	.40	.60	.70	.90			.0263	.0038	8.00	7
	.35	.00	.05	.30	.35	.40	.65	.70	.95			.0284	.0025	11.38	8
	.40	.00	.20	.20	.40	.60	.60	.80	1.00			.0300	.0025	12.00	5
	.45	.00	.00	.10	.35	.45	.55	.55	.90			.0309	.0038	8.25	6
	.50	.50	.50	.50	.50	1.00	1.00	1.00	1.00			.0313	.0000	-	2
9	.15	.00	.15	.15	.30	.45	.45	.60	.75	.90		.0142	.0028	5.04	7
	.20	.00	.20	.20	.40	.40	.60	.60	.80	1.00		.0178	.0020	9.00	5
	.25	.00	.00	.25	.25	.25	.50	.50	.75	.75		.0208	.0023	9.00	4
	.30	.00	.00	.20	.30	.30	.50	.60	.70	.90		.0233	.0026	9.00	7
	.35	.00	.00	.00	.30	.35	.35	.65	.65	.70		.0253	.0016	16.07	5
	.40	.00	.00	.00	.20	.40	.40	.60	.60	.80		.0267	.0030	9.02	5
	.45	.00	.10	.20	.30	.45	.55	.65	.75	.90		.0275	.0006	46.88	9
	.50	.50	.50	.50	.50	.50	1.00	1.00	1.00	1.00		.0278	.0031	9.00	2
10	.10	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	.0090	.0000	-	10
	.15	.00	.15	.15	.30	.40	.55	.55	.70	.85	1.00	.0128	.0025	5.10	7
	.20	.00	.20	.20	.40	.40	.60	.60	.80	.80	1.00	.0160	.0000	-	5
	.25	.00	.00	.25	.25	.50	.50	.50	.75	.75	1.00	.0188	.0025	7.50	4
	.30	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	.0210	.0000	-	10
	.35	.00	.05	.05	.35	.35	.40	.65	.70	.70	1.00	.0228	.0025	9.11	6
	.40	.00	.20	.20	.40	.40	.60	.60	.80	.80	1.00	.0240	.0000	-	5
	.45	.00	.00	.10	.10	[illegible]	.45	[illegible]	.55	.65	.65	.0245	[illegible]	[illegible]	[illegible]
	.50	.00	.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.50	1.00	.0250	.0000	-	2

† n^* - effective number of replications

Copy available to DDC does not
permit fully legible reproduction

The linear program solution for $N_1 = \ldots = N_n = N$ also reveals that the optimal $\underline{\theta}$ need not have distinct entries. For example, $n = 8$ and $p = .45$ gives $\theta_1 = \theta_2 = 0$, $\theta_3 = .10$, $\theta_4 = .35$, $\theta_5 = .45$, $\theta_6 = \theta_7 = .55$, $\theta_8 = .9$. Here one forms $\bar{R}_8$ as

$$\bar{R}_8 = (2R_1 + R_3 + R_4 + R_5 + 2R_6 + R_8)/8. \tag{22}$$

With regard to the actual simulation, one would use $R_2 = R_1$ and $R_7 = R_6$. These results imply that the suggested variance reduction technique can use no more than six replications for optimality. However, the weighting opportunities when $n = 8$ apparently allow a greater variance reduction 8.25 than in the case of $n = 6$, 7.07. Some additional cases, such as $n = 9$ and $p = .20$ also need clarification. Notice that $\theta_1 = 0$ and $\theta_9 = 1$. Since $W_9 = U + \theta_9 \pmod 1$ we have

$$\bar{R}_9 = (2R_1 + 2R_2 + 2R_4 + 2R_6 + R_8)/9 \tag{23}$$

Here five replications suffice for optimality.

Let us now summarize the properties of model 1.

1. The optimal $\underline{\theta}$ is independent of α and β. See (12).
2. For large $N_1, \ldots, N_n$, $\omega_n^{(1)}$ is independent of α and β. See (16).
3. For given p $\omega_n^{(1)}$ increases as $N_i \to N$ for $i = 1, \ldots, n$. See (16).
4. For equal N_i $\omega_n^{(1)} = (1 - q)/(1 - nq)$ where $q \equiv p \pmod{1/n}$. See Table 1.
5. For equal N_i the optimal solution may call for $n^* < n$ replications but with different weights for each observation. The result is a smaller variance and larger variance reduction than would occur for the optimal solution would yield for n^* replications with equal observation weights. See Table 1.

3. Model 2

The assumptions in model 1 that $X_1,\ldots, X_n$ be i.i.d., $Y_1,\ldots, Y_n$ be i.i.d. and that $\{X_i\}$ and $\{Y_i\}$ be independent are unnecessary restrictions whose removal can lead to an additional variance reduction. Assume that N_i is sufficiently large for $i=1,\ldots, n$ so that treating X_i and Y_i as normal variates with means $N_i\alpha$ and $N_i\beta$, respectively, and variances $N_i\alpha(1-\alpha)$ and $N_i\beta(1-\beta)$, respectively, introduces incidental error. Also assume that $W_1,\ldots, W_r$ follow (5). Let Z_i denote a random variable from the normal distribution with zero mean and unit variance, denoted by $N(0,1)$. Since either X_i <u>or</u> Y_i occur one can represent these quantities by

$$(24)\qquad \begin{aligned} X_i &= N_i\alpha + Z_i\sqrt{N_i\alpha(1-\alpha)} \\ Y_i &= N_i\beta + Z_i\sqrt{N_i\beta(1-\beta)}\ . \end{aligned}$$

Then (3) becomes for replication i

$$\Delta R_i = (\alpha-\beta)N_i\,\Delta I_i + A\sqrt{N_i}\;Z_i\,\Delta I_i + (Ap+B)\sqrt{N_i}\;Z_i$$

$$A \equiv \sqrt{\alpha(1-\alpha)} \qquad - B$$

$$B \equiv \sqrt{\beta(1-\beta)}$$

so that $\bar{R}_n$ has

$$(25)\qquad \mathrm{var}^{(2)}(\bar{R}_n) = (V_n^{(2)} + 2U_n^{(2)})/n$$

where $V_n^{(2)} = V_n^{(1)}$, as is expected, and

$$(26)\qquad nU_n^{(2)} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big\{\mathrm{cov}\,(I_i, I_j)\Big[N_i\,N_j\,(\alpha-\beta)^2 + A^2\sqrt{N_i\,N_j}\;\mathrm{corr}\,(Z_i, Z_j)\Big] + (Ap+B)^2\sqrt{N_i\,N_j}\;\mathrm{corr}\,(Z_i,Z_j)\Big\}\ .$$

If $Z_1,\ldots, Z_n$ are independent then

(27) $$nU_n^{(2)} = \sum_{i=1}^{n-1} N_i \sum_{j=i+1}^{n} N \operatorname{cov}(I_i, I_j)(\alpha-\beta)^2 .$$

However, it is possible to make $Z_1, \ldots, Z_n$ negatively correlated so that $\operatorname{corr}(Z_i, Z_j) < 0$ for all $i \neq j$. In particular, the most negative correlation achievable is†

(28) $$\operatorname{cov}(Z_i, Z_j) = -1/(n-1) \qquad i \neq j$$

so that

(29) $$nU_n^{(2)} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big\{ \operatorname{cov}(I_i, I_j)\,[(\alpha-\beta)^2 N_i N_j - A^2 \sqrt{N_i N_j}/(n-1)] - (Ap + B)^2 \sqrt{N_i N_j}/(n-1) \Big\} .$$

To induce (28) one proceeds as follows. Let $Z_1', \ldots, Z_n'$ be independent from $N(0, 1)$. Then form

(30) $$Z_i = \sum_{j=1}^{i} h_{ij} Z_j' \qquad i=1,\ldots, n$$

where one restricts the lower triangular matrix

$$\underline{h} = \begin{pmatrix} h_{11} & 0 \ldots & 0 \\ & & \vdots \\ h_{12} & h_{22} & \vdots \\ \vdots & \vdots & \ddots \\ h_{1n} & h_{2n} \cdots & h_{nn} \end{pmatrix}$$

so that

(31) $$\sum_{j=i}^{i} h^2_{ij} = 1$$
$$\sum_{k=1}^{i} h_{ik} h_{jk} = -1/(n-1) \qquad i<j, \quad i=1,\ldots, n .$$

In particular, solving (31) yields

$$h_{11} = 1$$

(32) $$h_{1j} = -1/(n-1) \qquad j = 2,\ldots, n$$

$$h_{ij} = h_{i+1,j} \qquad j = 2,\ldots i; \quad i = 2,\ldots, n-1.$$

† This is seen by noting that any smaller covariance makes the covariance matrix of $Z_1, \ldots, Z_n$ negative definite.

Algorithm VRN computes $\underline{h}$.

Algorithm VRN(n)

1. $\rho \leftarrow -1/(n-1)$.
2. $h_{11} \leftarrow 1$.
3. For $i=2(1)n$ $h_{i1} \leftarrow \rho$.
4. $j \leftarrow 2$.
5. $s \leftarrow 0$.
6. For $i=1(1)j-1$ $s \leftarrow s + h_{ji}^2$.
7. $h_{jj} \leftarrow \sqrt{1-s}$.
8. If $j = n$, deliver $\underline{h}$.
9. $s \leftarrow \rho$.
10. For $i = 1(1)j-1$ $s \leftarrow s-h_{ji}^2$.
11. $h_{j+1,j} \leftarrow s/h_{jj}$.
12. $j \leftarrow j + 1$.
13. If $j < n$, for $i = j+1(1)n$ $h_{i,j-1} \leftarrow h_{j,j-1}$.
14. Go to 5.

Consider the case for which $\mathrm{cov}(I_i, I_j) = 0$ for $i \neq j$. Then

(33) $$\omega_n^{(2)} = 1/[1 - 2 (Ap + B)^2 (n - 1)^{-1} \sum_{i=1}^{n-1} \sqrt{N_i} \sum_{j=i+1}^{n} \sqrt{N_j} / V_n^{(1)}]$$

is the variance reduction. Consider the case $N_1 = \ldots = N_n = N$. Then (33) is

(34) $$\omega_n^{(2)} = 1/\{1 - (Ap + B)^2/[p(1 - p)(\alpha - \beta)^2 N + p\alpha(1 - \alpha) + (1 - p)\beta(1 - \beta)]\}$$

Expressions (33) and (34) allow the observations:

1. For given α , β and p, variance reduction decreases as $N_1, \ldots, N_n$ increase.
2. For p close to one or zero

$$\omega_n^{(2)} \sim \sum_{i=1}^{n} N_i / [\sum_{i=1}^{n} N_i - 2(n - 1)^{-1} \sum_{i=1}^{n-1} \sqrt{N_i} \sum_{j=i+1}^{n} \sqrt{N_j}].$$

Clearly this model is of most benefit when the mixture is heavily weighted toward α or β, a result that is the converse of an observation for model 1 where small p leads to small variance reductions.

4. Model 3

The assumption of only two possible survival probabilities α and β in a given year is an abstraction usually forced on an investigator by incomplete knowledge. In a more realistic model one can conceive of the survival probability P being a continuous function of, say, rainfall F. For example, suppose that

$$P = e^{aF + b} \qquad a > 0, \quad b < 0 \tag{35}$$

where F is from $N(\mu,\sigma)$ and $pr(F \geq -b/a) \sim 0$. Here P increases with F as one might expect in an animal population where rainfall is positively correlated with the nutritional base available to the population.

Let subscripts on P and F denote replication number and assume that $F_i = \mu + Z_i\sigma$, where Z_i is from $N(0,1)$. If Z is formed as in (30) then $\text{corr}\,(F_i, F_j) = -1/(n-1)$ for n replications and one can show that

$$\begin{aligned} &E(P_i) = f(1,1) \\ &E(P_iP_j) = \begin{cases} f(4,2) & i = j \\ f(2-2/(n-1),2) & i \neq j \end{cases} \end{aligned} \tag{36}$$

$$f(\theta,\phi) \equiv e^{\theta a^2\sigma^2/2 + \phi(b + a\mu)}.$$

Let R_i denote the number of survivors out of a population of size N_i with a randomly selected survival probability P_i. Then

$$(37)\quad \begin{cases} E(R_i|P_i) = N_i\ P_i \\ E(R_i^2|P_i) = N_i\ P_i(1 - P_i) + (N_i\ P_i)^2 \\ E(R_i\ R_j|P_i,P_j) = N_i\ N_j\ P_i\ P_j & i \neq j \\ E(R_i) = f(1,1) \\ E(R_i^2) = N_i[f(1,1) - f(4,2) + N_i f(4,2)] \\ E(R_i\ R_j) = N_i\ N_j\ f(2 - 2/(n-1),\ 2) & i \neq j \end{cases}$$

so that

$$(38)\quad n^2\ \mathrm{var}^{(3)}(\bar{R}_n) = [f(4,2) - f(2,2)] \sum_{i=1}^{n} N_i^2 + [f(1,1) - f(4,2)] \sum_{i=1}^{n} N_i + 2[f(2 - 2/(n-1),\ 2) - f(2,2)] \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} N_i\ N_j .$$

Since

$$(39)\quad f(2 - 2/(n-1),\ 2) = f(2,\ 2)f(-\ 2/(n-1),\ 0)$$

the covariance contribution in (38) is negative.

Consider the case $N_1 = \ldots N_n = N$ for which the variance reduction is

$$(40)\quad \omega_n^{(3)} = 1/\{1 + (n-1)[f(-2/(n-1),0) - 1]/[f(2,0) - 1 + (f(-1,-1) - f(2,0))/N]\} .$$

For N large this is

$$(41)\quad \omega_n^{(3)} \sim 1/\{1 + (n-1)[f(-2/(n-1),\ 0) - 1]/[f(2,0) - 1]\} .$$

For given a and σ^2 (41) decreases monotonically as n increases, suggesting the appeal of correlating replications in pairs. Moreover, variance considerations indicate that for a total of n replications creating $n/2$ (n even) independent sets of replications each with two replications correlated according to model 3 yields a smaller variance than using model 3 to induce correlation among all n replications. Also, for a given n (41) monotonically decreases as a and σ^2 increase. Table 2 shows selected values of $\omega_2^{(3)}$. Notice that the greatest variance reduction accrues when P_i varies little between successive years.

Table 2

Selected Values of $\omega_2^{(3)}$

$a^2\sigma^2$	$\omega_2^{(3)}$	$a^2\sigma^2$	$\omega_2^{(3)}$	$a^2\sigma^2$	$\omega_2^{(3)}$	$a^2\sigma^2$	$\omega_2^{(3)}$
0.05	20.50	.30	3.86	.55	2.36	.80	1.82
.10	10.51	.35	3.39	.60	2.22	.85	1.75
.15	7.18	.40	3.03	.65	2.09	.90	1.69
.20	5.52	.45	2.76	.70	1.99	.95	1.63
.25	4.52	.50	2.54	.75	1.90	1.00	1.58

5. Example[†]

By way of illustration we describe the application of model 1 to a simulation of the population dynamics of a hypothetical female elephant population. Three attributes characterize each strata: age, maturity and pregnancy. Only mature elephant conceive and the gestation period is twenty-two months. Figure 1 shows how strata change from year to year. Each arc has an associated probability. In some cases this probability may be zero. For example, one year old calves have zero probability of maturing. Other probabilities are unity, as in the case of surviving pregnant elephants giving birth.

The remaining probabilities are functions of rainfall and density. In particular, survival, maturity and conception probabilities increase as rainfall increases but decrease as population density increases. For expository purposes the example neglects density dependence. However it does specify explicit relationships between the probabilities and rainfall. The variance reduction technique of model 1 was applied to the survival probabilities. Specifically

† This model emerged from discussions with Professor Daniel Botkin at the Ecosystems Center at the Marine Biological Laboratory at Woods Hole and with Professors Richard Miller and Matthew Sobel of Yale University.

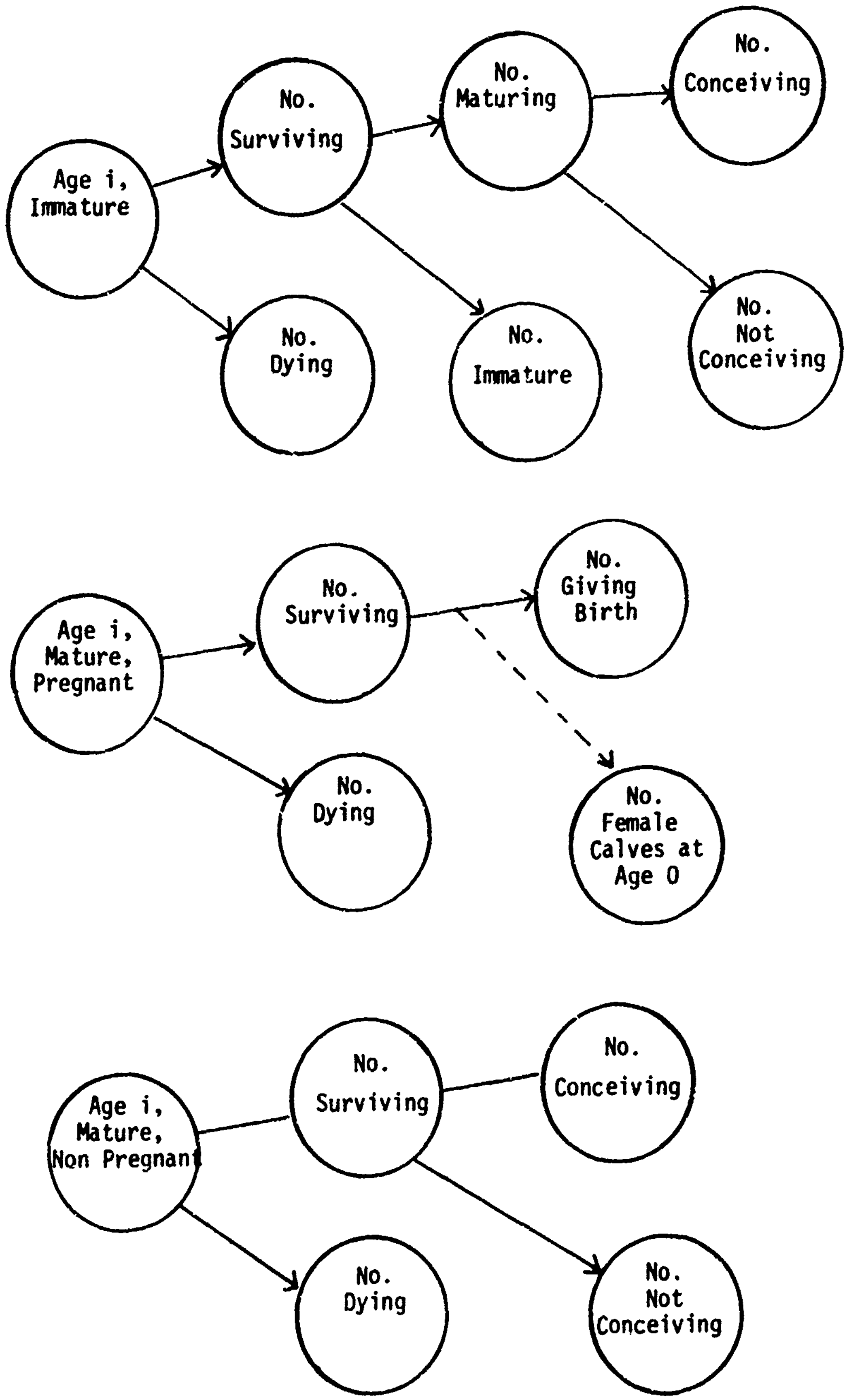


Fig. 1. Female Elephant Population Dynamics

let F denote rainfall in a given year and assume it has a normal distribution. Then the survival probability is α_k for strata k if $F \geq F^*$ and β_k if $F < F^*$, where $\alpha_k > \beta_k$. The quantity F^* is a threshold rainfall that determines survival prospects. In terms of model 1 $p = pr(F \geq F^*)$. In the present example $p = 0.25$.

At each point in time we performed 10 independent sets of sampling experiments for each strata, where each experiment consisted of $n=4$ replications. The independent sets enabled us to estimate the variance of the average over the 4 replications subjected to the variance reduction technique in each strata. For $p = .25$, $\theta_1 = 0$, $\theta_2 = .25$, $\theta_3 = .50$ and $\theta_4 = .75$ are an optimal solution. Each simulation began with the same population profile and ran for 100 years. For a given strata define R_{ij} as the number of surviving elephants on replication i of set j. Then

(43)
$$S = \frac{1}{40 \times 39} \sum_{i=1}^{4} \sum_{j=1}^{10} (R_{ij} - R_{..})^2,$$

where

(44)
$$R_{..} = \frac{1}{10} \sum_{j=1}^{10} R_{.j} \quad , \quad R_{.j} = \frac{1}{4} \sum_{i=1}^{4} R_{ij} \ ,$$

provides an estimate of the variance of $R_{..}$ if all replications were independent and

(45)
$$T = \frac{1}{10 \times 9} \sum_{j=1}^{10} (R_{.j} - R_{..})^2$$

provides an estimate of the variance of $R_{..}$ regardless of whether or not the replications are correlated. Then for strata k S_k/T_k provides a measure of variance reduction. Since sampling in strata k is independent of sampling in strata ℓ $(k \neq \ell)$ in a given year a summary measure of variance reduction is $(\Sigma S_k)/(\Sigma T_k)$.† Table 2 shows this ratio for years $m = 5(5)100$. In general,

† The only contrary case occurs between pregnant elephants that give birth and the new born calves. However, neglecting this correlation should not be a serious issue.

the results there suggest a halving of the cost needed to obtain the resulting accuracy.

Table 2

Estimated Variance Reduction in
Female Elephant Population Simulation†

year	V.R.	year	V.R.	year	V.R.	year	V.R.
5	1.73	30	2.03	55	2.17	80	2.22
10	1.43	35	2.24	60	2.15	85	1.97
15	1.82	40	2.05	65	2.20	90	1.84
20	1.85	45	2.44	70	2.10	95	1.85
25	2.12	50	2.18	75	2.51	100	2.14

† V.R. = Variance Reduction

6 References

1. Fishman, G.S., Concepts and Methods in Discrete Event Digital Simulation, John Wiley and Sons, 1973.

2. Hammersley, J.M. and Handscomb, D.C., Monte Carlo Methods, Methuen, London, 1964.

3. Kleijnen, J.P.C., Statistical Techniques in Simulation, Vols. 1 and 2, Marcel Dekker, New York, 1975.

4. Mihram, A., Simulation: Statistical Foundations and Methodology, Academic Press, 1972.